Aggregating Slow and Fast Growth

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March 2026

Abstract

In this note I consider an economy with two factor inputs, one growing exponentially and one growing hyperbolically. I derive the conditions under which aggregate output is growing super exponentially vs subexponentially. In the case where the slow growing factor is in fact fixed, I get a closed form solution for the date at which the growth regime changes from super to subexponential.

1 Introduction

Suppose part of an economy grows at a hyperbolic rate—growing to infinity in finite time—while the rest of the economy grows at a merely exponential rate. What is the aggregate growth path of the economy? Aghion et al. (2019) derive the conditions on a growth environment that induces hyperbolic growth and derive a closed form solution for the 'singularity date' of that hyperbolically growing factor. Here I am interested in a slightly weaker condition: how long can an economy with slow and fast growing factors sustain *superexponential* growth (the growth rate is growing over time)? And analogously to the singularity date in Aghion et al. (2019), what is the 'growth regime change date' where the economy tips from super to *subexponential* growth (aggregate growth slowing over time)?

Below I show that one can define a sufficient condition on two variables (elasticities of substitution between factor inputs and the term dictating speed of growth of the hyperbolic variable) that guarantees the growth environment can sustain superexponential growth of aggregate output right up to the one-factor-singularity. I also derive a closed form solution for the date at which the regime tips from super to subexponential growth in the case where this sufficient condition is not met.

Part of the motivation for this note is the observation of Davidson et al. (2026) that surprisingly weak conditions on automation are sufficient to tip an economy into a hyperbolic growth regime. This result is built on a number of simplifying assumptions, the most consequential being that aggregation of human and AI-equivalent labor is Cobb-Douglas. This means that aggregate superexponential growth can be sustained by fast growth in a small sector of the economy since slow growing factors do not bottleneck aggregate growth in such an environment. Here I show that if growth in a single sector is fast enough, that can be sufficient for the aggregate economy to sustain superexponential growth. Importantly, that condition is independent of the size of the contribution of that sector to output. In such a case, Cobb-Douglas aggregation becomes a more reasonable simplifying assumption for the near term growth implications of automation.

2 Simple Model

2.1 Factor and Aggregate Growth

Suppose we have the aggregate output function

$$Y_t = \left((1-\theta)^{\frac{1}{\sigma}} S_t^{\frac{\sigma-1}{\sigma}} + \theta^{\frac{1}{\sigma}} F_t^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{1}$$

where S is the size of the slow growing factor and F is the size of the fast growing factor, and $\theta \in (0,1)$ is the fast growing share of the economy. I assume $\sigma \leq 1$ so that the slow growing and fast growing factor are complements.

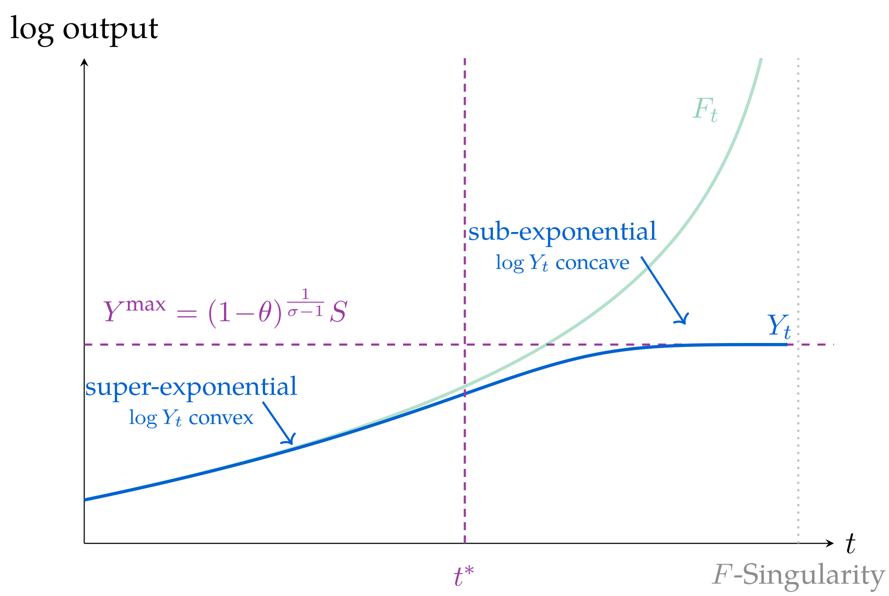
Suppose that the laws of motion for each of the slow and fast factors are given by

$$\text{(Exponential)} \qquad \dot{S}_t = g_S S_t \tag{2}$$

$$\text{(Hyperbolic)} \qquad \dot{F}_t = \alpha F_t^{1+\nu} \tag{3}$$

where $g_S \geq 0$ so S is growing exponentially and $\nu > 0$ so F is growing hyperbolically, and $\alpha > 0$. So our question here is, what are the conditions on θ, σ, g_S, and ν such that the growth rate of Y is increasing over time, and how long might that period last before slow growth of S stifels aggregate growth? Figure 1 illustrates the variables of interest in an environment where S is fixed over time, so that output converges to a constant.

Figure 1: Growth Regime Change at t^* $(g_S = 0)$



2.2 The Key Elasticity

The key term that is going to pin down the date where this system transitions from super to subexponential growth in Y is the elasticity

$$\varepsilon_t \equiv \frac{\partial \log Y_t}{\partial \log F_t} = \frac{\left(\frac{F_t}{S_t}\right)^{\frac{\sigma-1}{\sigma}}}{\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{\sigma}} + \left(\frac{F_t}{S_t}\right)^{\frac{\sigma-1}{\sigma}}} \tag{4}$$

so if F_t changes by 1%, Y_t changes by ε%. We can use this to simplify the expression for the growth rate of Y_t:

$$g_{Y,t} = \varepsilon_t g_{F,t} + (1 - \varepsilon_t) g_S \tag{5}$$

Then taking time derivatives we have

$$\dot{g}_{Y,t} = \dot{\varepsilon}_t (g_{F_t} - g_S) + \varepsilon_t \dot{g}_{F_t} \tag{6}$$

and the regime tips from super to subexponential at the time t^* such that $\dot{g}_{Y,t^*} = 0$. That is, when

$$-\dot{\varepsilon}_{t^*}(g_{F,t^*} - g_S) = \varepsilon_{t^*}\dot{g}_{F,t^*} \tag{7}$$

Note, that since $g_{F,t} > g_S$ (eventually) then the impact of F on Y is declining over time. That is, $\dot{\varepsilon}_t$ must be negative (eventually). Further, given our expression for ε in 4 we can derive the growth rate of the elasticity:

$$\frac{\dot{\varepsilon}_t}{\varepsilon_t} = \frac{\sigma - 1}{\sigma}(1 - \varepsilon_t)(g_{F,t} - g_S) \tag{8}$$

and given our expression for the hyperbolically growing variable, in equation (3) we have

$$g_{F,t} = \alpha F_t^{\nu} \implies \frac{\dot{g}_{F,t}}{g_{F,t}} = \nu g_{F,t} \tag{9}$$

Combining equations (7), (8), and (9) we have the condition

$$\varepsilon_{t^*} = 1 - \frac{\nu\sigma}{1-\sigma}\left(\frac{g_{F,t^*}}{g_{F,t^*} - g_S}\right)^2 \tag{10}$$

That is, when the output-F elasticity is greater than the right hand side of (10) then the regime is superexponential, and lower, it is subexponential. Though, noting that both sides of this equation are changing over time so we can't characterize a clean closed form for the t^* that pins down this equality.

We can further extend this to ensure that the starting conditions of the model are such that there is initial superexponential growth. Namely,

Condition (Initial superexponential growth)**.**

$$\varepsilon_0 > 1 - \frac{\nu\sigma}{1-\sigma}\left(\frac{F_0^{\nu}}{F_0^{\nu} - g_S}\right)^2 \tag{11}$$

2.3 No Growth S Case

We can see that if $g_S = 0$ then we can collapse this elasticity condition to

$$\varepsilon_{t^*} = 1 - \frac{\nu\sigma}{1-\sigma}$$

and therefore, using our expression for the key elasticity, we can back out the regime change ratio as a function of parameters:

$$\frac{F_{t^*}}{S} = \left(\frac{1-\theta}{\theta}\right)^{\frac{-1}{1-\sigma}} \left(\frac{1-\sigma(1+\nu)}{\nu\sigma}\right)^{\frac{-\sigma}{1-\sigma}}. \tag{12}$$

This gives us a specific threshold value of F such that the growth regime tips. We can further characterize this by solving the differential equation that pins down the growth rate of F, to

$$F_t = [\nu\alpha(T-t)]^{\frac{-1}{\nu}} \tag{13}$$

where T is the singularity date of F. Therefore, we get the closed form solution for the date of the growth regime change

$$t^* = T - \frac{1}{\nu\alpha}\left[\left(\frac{1-\theta}{\theta}\right)^{\frac{1}{1-\sigma}} \left(\frac{1-\sigma(1+\nu)}{\nu\sigma}\right)^{\frac{\sigma}{1-\sigma}}\right]^{\nu} \tag{14}$$

where the second term is difference between the F-singularity date and the growth regime change date. We can see that when F make up a larger share of the economy (θ close to 1) then the difference between the singularity and regime change time shrinks, meaning that the economy sustains superexponential growth right up until the F-singularity. Further, if the numerator of the second fraction is negative ($\sigma(1+\nu) > 1$) then we likewise have superexponential growth is sustained up to the singularity. From this observation we have the result:

Result. *The growth system defined above (equations (1), (2), and (3)) can sustain superexponential growth up until the F-singularity date so long as there is initial superexponential growth ((11) is satisfied) and*

$$\sigma > \frac{1}{1+\nu}.$$

This result generalizes this observation from the no growth S case to the case of any choice of g_S ¿ 0 since positive growth in S only increases the growth rate of aggregate output. Thus we can see that high substitutability and faster growing F (higher ν) can make sustained superexponential growth more feasible.

3 Calibration Examples

Here I pin down the growth system by imposing assumptions on parameters and starting conditions of the model described above. In particular, I assume

- Starting from 2025, the F-singularity date is 2100, so $T - t$ is 75;
- $\alpha = \frac{1}{\nu T}$, so that $F_t = [(T-t)/T]^{\frac{-1}{\nu}}$;
- $\theta \in \{0.2, 0.5\}$ and $\sigma \in \{0.3, 0.5, 0.7\}$;
- $F_0 = S_0 = 1$ and;
- $g_{Y,0} = 2.5\%$ and $g_S = 2\%$.

Figure 2 illustrates the time path of aggregate output and factors (normalized by starting values) as well as growth rates.

As we can see from this figure there are cases where path of aggregate output is super and subexponential for all years before the F-singularity (namely, $\theta = 0.5$, $\sigma = 0.7$; and $\theta = 0.2$, $\sigma = 0.3$ respectively). This falls out directly of the analytical conditions above.

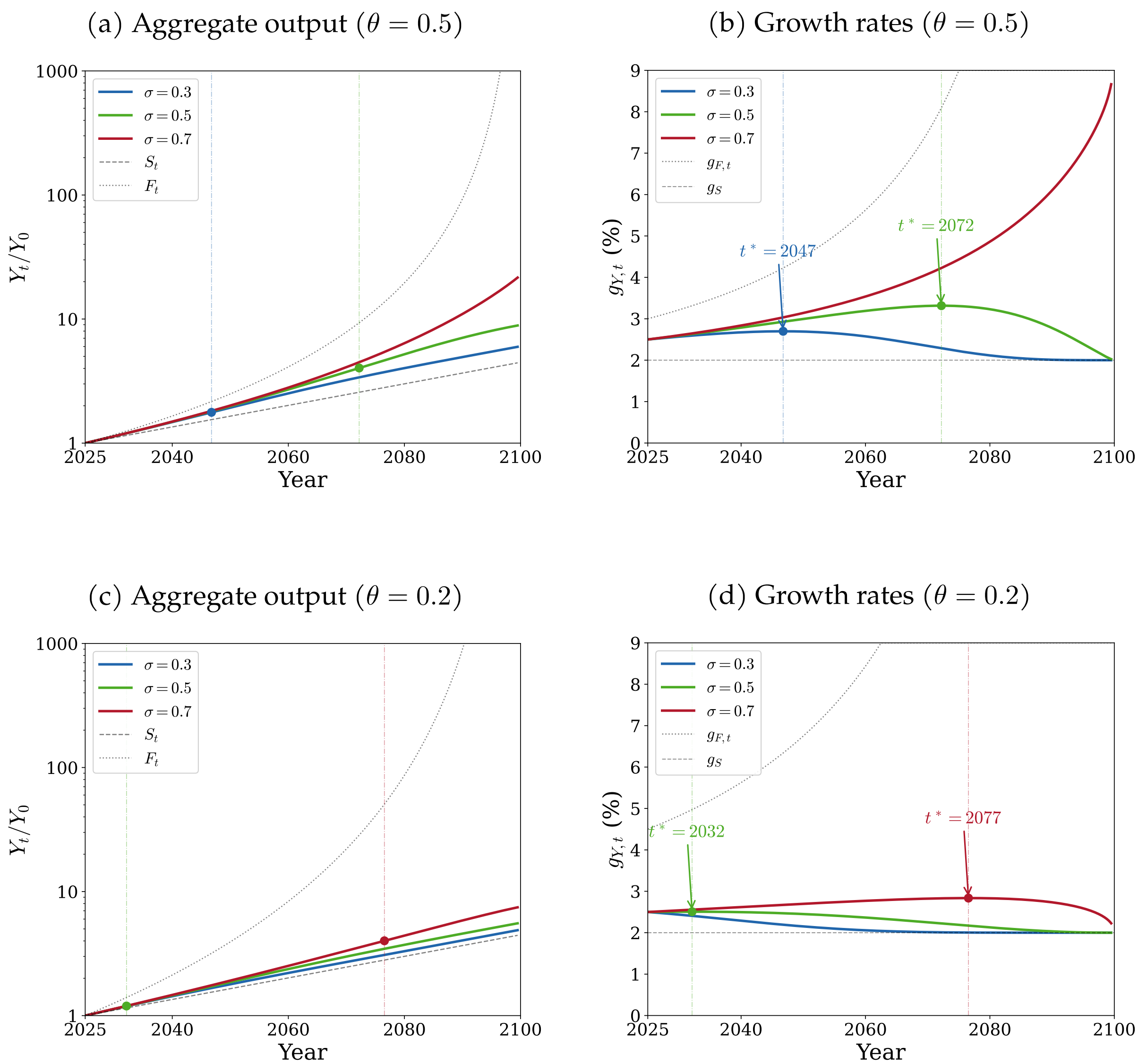
In the case where $\theta = 0.5$, since factors are starting at the same levels, $\varepsilon_0 = 0.5$. Therefore we can use equation (5) to infer ν:

$$
\begin{aligned}
g_{Y,0} &= \varepsilon_0 g_{F,0} + (1-\varepsilon_0) g_S \\
\implies 0.025 &= 0.5 \times g_{F,0} + 0.5 \times 0.02 \\
\implies g_{F,0} &= 0.03
\end{aligned}
$$

then given $F_0 = 1$, from the law of motion for F in equation (3) we have $\alpha = 0.03$ and therefore, $\nu = 1/(\alpha T) = 1/(0.03 \times 75) = 4/9$. Therefore, we can apply the Result above, so we know the system will maintain superexponential growth so long as $\sigma > 1/(1+\nu) = 9/13 \approx 0.69$. Thus, we can see why a calibration of $\sigma = 0.7$ demonstrates sustained superexponential growth.

Likewise, we could also apply the Condition above to verify that under $\sigma = 0.3$ and $\theta = 0.2$ the inequality (11) is not satisfied, thus the system never reaches superexponential growth.

Figure 2: Calibrated output paths and growth rates ($g_S = 2\%$, $g_{Y,0} = 2.5\%$, $T = 75$)



4 Conclusion

The results above demonstrate that rapidly accelerating progress in a subset of the economy can sustain increasing aggregate growth rates. It also demonstrates that such a rapidly accelerating sector can totally fail to sustain higher aggregate growth rates. Unsurprisingly, these results depend crucially on the output share of that rapidly growing factor and the complementarity of factor inputs in production.

The figures above plot aggregate growth rates only up to the F-singularity date. For calibrations where growth tips before the singularity, this is without loss: aggregate growth is already converging to g_S by that point. For calibrations that sustain superexponential growth up to the singularity, however, the transition is abrupt. Once F reaches infinity, the elasticity ε drops to zero, and the growth rate of output jumps discontinuously from a potentially very large value down to g_S.

References

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